7 Binomial Theorem

The power of the binomial theorem is used to generate and distribute IP addresses to an ever-increasing number of electronic devices, which have increased significantly in the past decade thanks to the rise in industries like the Internet of Things (IoT).

Topic Notes

- *Basics of Binomial Theorem*

BASICS OF BINOMIAL THEOREM 1

TOPIC 1

BINOMIAL THEOREM FOR POSITIVE INTEGER

Introduction

In mathematics, an algebraic expression is an expression built up from constants, variables, and a finite number of algebraic operations (like addition, subtraction, etc). An algebraic expression that contains only two terms is called a binomial. In this chapter, we shall discover the answer to the following question:

"What happens when you multiply a binomial by itself (as many times as you want)?"

In other words, we want to expand $(x + y)^n$ for any positive integral index n. Before proceeding further, let us first understand a beautiful pattern of numbers known as 'Pascal's Triangle'.

Pascal's Triangle

Pascal's triangle, in algebra, a triangular arrangement of numbers that gives the coefficients in the expansion of any binomial expression, such as $(x + y)^n$. It is named for the 17th-century French mathematician Blaise Pascal.

```
                    1
                 1     1
              1     2     1
           1     3     3     1
        1     4     6     4     1
     1     5    10    10     5     1
  1     6    15    20    15     6     1
1    7    21    35    35    21     7     1
```

Binomial Theorem for Positive Integral Index

The Binomial Theorem helps us to expand any positive integral power of the binomial expression. It was first given by English physicist and mathematician Issac Newton. The Binomial Theorem for positive integral index can be states as follows:

For every positive integer n, we have

$$(x+y)^n = \sum_{r=0}^{n} {}^nC_r x^{n-r} y^r$$

$$= {}^nC_0 x^n y^0 + {}^nC_1 x^{n-1} y^1 + \ldots + {}^nC_{n-1} x^1 y^{n-1} + {}^nC_n x^0 y^n.$$

Binomial Theorem for any Positive Integer n

Binomial theorem states that for any given positive integer n, the expression of the n^{th} power of the sum of any two numbers a and b may take place as the sum of $n + 1$ terms of the particular form

$$(x+y)^n = \sum_{r=0}^{n} {}^nC_r x^{n-r} y^r$$

$$= {}^nC_0 x^n y^0 + {}^nC_1 x^{n-1} y^1 + \ldots + {}^nC_{n-1} x^1 y^{n-1} + {}^nC_n x^0 y^n.$$

Binomial Theorem for Positive Integral Indices

Let us now observe the following identities:

$$(x + y)^0 = 1$$
$$(x + y)^1 = x + y$$
$$(x + y)^2 = x^2 + 2xy + y^2$$
$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

One can easily observe that

(i) The exponent of x in the expansion of $(x + y)^n$ is decreasing by 1 from n to 0.

(ii) The exponent of y in the expansion of $(x + y)^n$ is increasing by 1 from 0 to n.

(iii) The sum of exponents of x and y in the expansion of $(x + y)^n$ is always n.

(iv) The number of terms in the expansion of $(x + y)^n$ is always $n + 1$.

(v) The coefficients of terms in above expression are actually making Pascal's Triangle.

So, using Pascal's triangle one can easily write the expansion of $(x + y)^5$, $(x + y)^6$ and other exponents as follows:

$$(x + y)^5 = x^5 + 5x^4 y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

$$(x + y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$$

But, working with Pascal's Triangle can be very time-consuming if the exponent is a very large number (say, 100). So, we require a formula to find the value at any place in Pascal's Triangle.

The value at $(n + 1)^{th}$ row and $(r + 1)^{th}$ place of Pascal's Triangle is

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

For example, the value at the 5^{th} row and 3^{rd} place is

$${}^4C_2 = \frac{4!}{2!(4-2)!} = 6.$$

Example 1.1: Expand the expression $(1 - 2x)^5$. [NCERT]

Ans. By using Binomial Theorem, the expression $(1 - 2x)^5$ can be expanded as

$(1 - 2x)^5$

$= {}^5C_0(1)^5 - {}^5C_1(1)^4(2x) + {}^5C_2(1)^3(2x)^2 - {}^5C_3(1)^2(2x)^3 + {}^5C_4(1)^1(2x)^4 - {}^5C_5(2x)^5$

$= 1 - 5(2x) + 10(4x^2) - 10(8x^3) + 5(16x^4) - (32x^5)$

$= 1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5$

which is the required expansion.

Example 1.2: Expand the expression $\left(\frac{2}{x}-\frac{x}{2}\right)^5$. [NCERT]

Ans. By using Binomial Theorem, the expression $\left(\frac{2}{x}-\frac{x}{2}\right)^5$ can be expanded as

$$\left(\frac{2}{x}-\frac{x}{2}\right)^5 = {}^5C_0\left(\frac{2}{x}\right)^5 - {}^5C_1\left(\frac{2}{x}\right)^4\left(\frac{x}{2}\right) + {}^5C_2\left(\frac{2}{x}\right)^3\left(\frac{x}{2}\right)^2 - {}^5C_3\left(\frac{2}{x}\right)^2\left(\frac{x}{2}\right)^3 + {}^5C_4\left(\frac{2}{x}\right)\left(\frac{x}{2}\right)^4 - {}^5C_5\left(\frac{x}{2}\right)^5$$

$$= \frac{32}{x^5} - 5\left(\frac{16}{x^4}\right)\left(\frac{x}{2}\right) + 10\left(\frac{8}{x^3}\right)\left(\frac{x^2}{4}\right) - 10\left(\frac{4}{x^2}\right)\left(\frac{x^3}{8}\right) + 5\left(\frac{2}{x}\right)\left(\frac{x^4}{16}\right) - \frac{x^5}{32}$$

$$= \frac{32}{x^5} - \frac{40}{x^3} + \frac{20}{x} - 5x + \frac{5}{8}x^3 - \frac{x^5}{32}$$

Example 1.3: Evaluate $\left(\sqrt{3}+\sqrt{2}\right)^6 - \left(\sqrt{3}-\sqrt{2}\right)^6$. [NCERT]

Ans. Firstly, the expression $(a + b)^6 - (a - b)^6$ is simplified by using Binomial Theorem. This can be done as,

$$(a+b)^6 = {}^6C_0a^6 + {}^6C_1a^5b + {}^6C_2a^4b^2 + {}^6C_3a^3b^3 + {}^6C_4a^2b^4 + {}^6C_5a^1b^5 + {}^6C_6b^6$$

$$= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

$$(a-b)^6 = {}^6C_0a^6 - {}^6C_1a^5b + {}^6C_2a^4b^2 - {}^6C_3a^3b^3 + {}^6C_4a^2b^4 - {}^6C_5a^1b^5 + {}^6C_6b^6$$

$$= a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6$$

$\therefore\ (a+b)^6 - (a-b)^6 = 2[6a^5b + 20a^3b^3 + 6ab^5]$

Putting $a = \sqrt{3}$ and $b = \sqrt{2}$, we obtain

$$\left(\sqrt{3}+\sqrt{2}\right)^6 - \left(\sqrt{3}-\sqrt{2}\right)^6$$

$$= 2\left[6\left(\sqrt{3}\right)^5\left(\sqrt{2}\right) + 20\left(\sqrt{3}\right)^3\left(\sqrt{2}\right)^3 + 6\left(\sqrt{3}\right)\left(\sqrt{2}\right)^5\right]$$

$$= 2[54\sqrt{6} + 120\sqrt{6} + 24\sqrt{6}]$$

$$= 2 \times 198\sqrt{6}$$

$$= 396\sqrt{6}$$

Example 1.4: Prove that $\sum_{r=0}^{n} 3^r\ {}^nC_r = 4^n$. [NCERT]

Ans. By Binomial Theorem,

$$\sum_{r=0}^{n} {}^nC_r\, a^{n-r}b^r = (a+b)^n$$

By putting $b = 3$ and $a = 1$ in the above equation, we obtain

$$\sum_{r=0}^{n} {}^nC_r(1)^{n-r}(3)^r = (1+3)^n$$

$$\Rightarrow \quad \sum_{r=0}^{n} 3^r\ {}^nC_r = 4^n$$

Hence, proved.

Example 1.5: Show that $9^{n+1} - 8n - 9$ is divisible by 64, whenever n is a positive integer. [NCERT]

Ans. In order to show that $9^{n+1} - 8n - 9$ is divisible by 64, it has to be proved that,

$9^{n+1} - 8n - 9 = 64k$, where k is some natural number

By Binomial Theorem,

$$(1 + 8)^{n+1} = {}^{n+1}C_0 + {}^{n+1}C_1(8) + {}^{n+1}C_2(8)^2 + \ldots + {}^{n+1}C_{n+1}(8)^{n+1}$$

$$\Rightarrow 9^{n+1} = 1 + (n + 1)(8) + 8^2\,[{}^{n+1}C_2 + {}^{n+1}C_3 \times 8 + \ldots + {}^{n+1}C_{n+1}(8)^{n-1}]$$

$$\Rightarrow 9^{n+1} = 9 + 8n + 64[{}^{n+1}C_2 + {}^{n+1}C_3 \times 8 + \ldots + {}^{n+1}C_{n+1}(8)^{n-1}]$$

$\Rightarrow 9^{n+1} - 8n - 9 = 64k$, where $k = {}^{n+1}C_2 + {}^{n+1}C_3 \times 8 + \ldots + {}^{n+1}C_{n+1}(8)^{n-1}$

is a natural number.

Thus, $9^{n+1} - 8n - 9$ is divisible by 64, whenever n is a positive integer.

OBJECTIVE Type Questions

[1 mark]

Multiple Choice Questions

1. **If P and Q are the coefficient of a^r and a^{n-r} respectively in the expansion of $(1 + a)^n$, then:**

(a) $P = Q$ (b) $P \neq Q$

(c) $P = \lambda Q$ for some λ (d) none of these

Ans. (a) $P = Q$

Explanation: We have,

P = coefficient of a^r in the expansion of

$(1 + a)^n = {}^nC_r$

Q = coefficient of a^{n-r} in the expansion of

$(1 + a)^n = {}^nC_{n-r}$

Now, ${}^nC_r = {}^nC_{n-r}$

$\Rightarrow P = Q.$

2. **The number of terms in the expansion of $(4 + 4x + x^2)^{20}$, when expanded in descending powers of x, is:**

(a) 20 (b) 21

(c) 40 (d) 41

Ans. (d) 41

Explanation: We have,

$$(4 + 4x + x^2)^{20} = [(2 + x)^2]^{20} = (2 + x)^{40}.$$

Therefore, there are 41 terms in the expansion of $(4 + 4x + x^2)^{20}$.

3. **The total number of terms in expansion of $(x + a)^{100} + (x - a)^{100}$ after simplification is:**

(a) 202 (b) 51

(c) 50 (d) none of these

[Delhi Gov. QB 2022]

Ans. (b) 51

Explanation: In the above binomial expansion, the terms at the even places will get eliminated, and we would be left with twice the sum of the terms at odd places.

Hence there will be

$$\frac{n}{2} = \frac{100}{2} + 1$$

= 51 terms

4. **The largest coefficient in the expansion of $(a + b)^{18}$ is:**

(a) ${}^{18}C_{18}$ (b) ${}^{18}C_{12}$

(c) ${}^{18}C_9$ (d) ${}^{18}C_6$

Ans. (c) ${}^{18}C_9$

Explanation: We know that, if n is even then ${}^nC_r$ is greatest for $r = \frac{n}{2}$. Therefore ${}^{18}C_r$ is greatest for $r = 9$.

Hence, the greatest coefficient is ${}^{18}C_9$.

5. **The number of terms in the expansion of $(1 + q)^9 + (1 - q)^9$ is:**

(a) 5 (b) 7

(c) 9 (d) 10

Ans. (a) 5

Explanation: We have,

$(p + q)^9 + (p - q)^9$

$= 2({}^9C_0 + {}^9C_2 (q)^2 + \ldots + {}^9C_8 (q)^8)$

Thus, it has 5 terms.

6. **The total number of terms in the expansion of $(x + a)^{100} + (x - a)^{100}$ after simplification is:**

(a) 50 (b) 202

(c) 51 (d) none of these

[NCERT Exemplar]

Ans. (c) 51

Explanation: Given,

$(x + a)^{100} + (x - a)^{100}$

$= ({}^{100}C_0 x^{100} + {}^{100}C_1 x^{99} a + {}^{100}C_2 x^{98} a^2 + \ldots) + ({}^{100}C_0 x^{100} - {}^{100}C_1 x^{99} a + {}^{100}C_2 x^{98} a^2 + \ldots)$

$= 2({}^{100}C_0 x^{100} + {}^{100}C_2 x^{98} a^2 + \ldots + {}^{100}C_{100} a^{100})$

So, there are 51 terms.

7. **Given the integers $r > 1$, $n > 2$, and coefficient of $(3r)^{th}$ and $(r + 2)^{nd}$ terms in the binomial expansion of $(1 + x)^{2n}$ are equal, then:**

(a) $n = 2r$ (b) $n = 3r$

(c) $n = 2r + 1$ (d) none of these

[NCERT Exemplar]

Ans. (a) $n = 2r$

Explanation: Given $(1 + x)^{2n}$

$$T_{3r} = T_{(3r-1)+1} = {}^{2n}C_{3r-1} x^{3r-1}$$

and $T_{r+2} = T_{(r+1)+1} = {}^{2n}C_{r+1} x^{r+1}$

Given, ${}^{2n}C_{3r-1} = {}^{2n}C_{r+1}$

$$3r - 1 + r + 1 = 2n$$

$[\because {}^nC_x = {}^nC_y \Rightarrow x + y = n]$

Or, $n = 2r$

Assertion Reason Questions

Direction: In the following questions, a statement of Assertion (A) is followed by a statement of Reason (R).

Choose the correct answer out of thefollowing choices.

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).

(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).

(c) (A) is true but (R) is false.

(d) (A) is false but (R) is true.

8. Assertion (A): The coefficient of a^4b^5 in the expansion of $(a + b)^9$ is ${}^9C_4$.

Reason (R): The formula of $(a + b)^n$ is ${}^nC_0a^nb^0 + {}^nC_1a^{n-1}b^1 + \ldots\ldots\ldots {}^nC_n a^n$.

Ans. (a) *Both (A) and (R) are true and (R) is the correct explanation of (A).*

Explanation: We know that,

$$(a + b)^n = {}^nC_0a^nb^0 + {}^nC_1a^{n-1}b^1 + \ldots {}^nC_na^n.$$

Now, a^4b^5 occurs in 5^{th} term of $(a + b)^9$, therefore coefficient of a^4b^5 is ${}^9C_4$.

9. Assertion (A): Let x be a true integer. If the coefficient of 2^{nd}, 3^{rd} and 4^{th} term of expansion $(1 + x)^3$ are in A.P then the value of x is 7.

Reason (R): The common difference of A.P. are different.

Ans. (c) *(A) is true but (R) is false.*

Explanation: As ${}^xC_1$, ${}^xC_2$ and ${}^xC_3$ are in A.P

$$\Rightarrow \quad 2{}^xC_2 = {}^nC_1 + {}^nC_3$$

$$x(x-1) = x + \frac{(x)(x-1)(x-2)}{6}$$

$$x - 1 = \frac{1+(x-1)(x-2)}{6}$$

$$6x - 6 = 6 + x^2 - 3x + 2$$

$$x^2 - 9x + 14 = 0$$

$$(n - 2)(n - 7) = 0$$

$$n = 2 \text{ or } 7$$

The common difference of an A.P is always same.

10. Assertion (A): The sum of the last eight coefficients in the expansion of $a(1 + x)^{16}$ is 2^{15}.

Reason (R): If x is an odd integer, then $({}^2C_0 - {}^2C_1 - 1{}^2C_2 - {}^2C_3 + \ldots + (-1))^{x}\,{}^2C_x \neq 0$.

Ans. (c) *(A) is true but (R) is false.*

Explanation: We have,

$${}^{16}C_0 + {}^{16}C_1 + \ldots + {}^{16}C_{16} = 2^{16}$$

[putting $x = 1$]

$$\Rightarrow \quad 2({}^{16}C_8 + {}^{16}C_9 + \ldots + {}^{16}C_{16}) = 2^{16}$$

$$\Rightarrow \quad {}^{16}C_8 + {}^{16}C_9 + \ldots + {}^{16}C_{16} = 2^{15}$$

$${}^2C_0 - {}^2C_1 + {}^2C_2 - {}^2C_3 + \ldots (-1)^x\,{}^2C_x = 0$$

CASE BASED Questions (CBQs)

[4 & 5 marks]

Read the following passages and answer the questions that follow:

11. *Ms. Khushi and Mr. Daksh decide to construct a Pascal triangle with the help of binomial theorem. They use the formula for the expansion is*

$$(x+y)^n = \sum_{r=0}^{n} {}^nC_r x^{n-r} y^r$$

$$= {}^nC_0 x^n y^0 + {}^nC_1 x^{n-1} y^1 + \ldots + {}^nC_{n-1} x^1 y^{n-1} + {}^nC_n x^0 y^n.$$

(A) The coefficient of x^k $(0 \le k \le n)$ in the expansion of $E = 1 + (1 + x) + (1 + x)^2 + \ldots (1 + x)^n$ is:

(a) ${}^{n+1}C_{k+1}$ (b) ${}^nC_k$

(c) ${}^{n+1}C_{n-k-1}$ (d) none of these

(B) The coefficient of y is the expansion of $\left(y^2 + \frac{c}{y}\right)^5$ is:

(a) $10\,c^3$ (b) $20\,c^2$

(c) $10\,c$ (d) $20\,c$

(C) The number of terms in the expansion of $(1+\sqrt{5}x)^7 + (1-\sqrt{5}x)^7$ are:

(a) 4 (b) 8

(c) 5 (d) 9

(D) The sum of coefficient of even powers x in the expansion of $\left(x - \frac{1}{x}\right)^{2n}$ is:

(a) $11 \times {}^{11}C_5$ (b) $\frac{11}{2} \times {}^{11}C_6$

(c) $11({}^{11}C_5 + {}^{11}C_6)$ (d) 0

(E) Assertion (A): The value of $(101)^4$ using the binomial theorem is 104060401.

Reason (R): $(x + y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}.y + {}^nC_2x^{n-2}.y^2 + ... + {}^nC_ny^n$.

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).
(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).
(c) (A) is true but (R) is false.
(d) (A) is false but (R) is true.

Ans. (A) *(a)* ${}^{n+1}C_{k+1}$

Explanation:

$$E = \frac{(1+x)^{n+1} - 1}{(1+x) - 1}$$

$$= \frac{{}^{n+1}C_0 + {}^{n+1}C_1x + {}^{n+1}C_2x^2 + \ldots - 1}{x}$$

$$= {}^{n+1}C_1 + {}^{n+1}C_2\,x + {}^{n+1}C_3\,x^2 + \ldots$$

Coefficient of $x^4 = {}^{n+1}C_{k+1}$

(B) *(a)* $10\,c^3$

Explanation:

$$\left(y^2 + \frac{c}{y}\right)^5 = {}^5C_0\left(\frac{c}{y}\right)^0 (y^2)^{5-0} + {}^5C_1\left(\frac{c}{y}\right)^1 (y^2)^{5-1} + \ldots + {}^5C_5\left(\frac{c}{y}\right)^5 (y^2)^{5-5}$$

$$= \sum_{r=0}^{5} {}^5C_r\left(\frac{c}{y}\right)^r (y^2)^{5-r}$$

We need coeffcient of y

$\Rightarrow \quad 2(5 - r) - r = 1$

$\Rightarrow \quad 10 - 3r = 1$

$\Rightarrow \quad r = 3$

So, coefficient of $y = {}^5C_3 . c^3$

$= 10c^3$

(C) *(a)* 4

Explanation: Given expansion is

$$\left(1+\sqrt{5}x\right)^7 + \left(1-\sqrt{5}x\right)^7.$$

Here, $n = 7$, which is odd.

Total number of terms $= \frac{n+1}{2}$

$= \frac{7+1}{2}$

$= \frac{8}{2}$

$= 4$

(D) *(d)* 0

Explanation: $(r + 1)^{\text{th}}$ term $= {}^{11}C_r\,(x)^{11-r}.\,x^{-r}$

$= {}^{11}C_r\,.\,x^{11-2r}$

Even power of x exists only if $11 - 2r =$ an even number which is not possible

Thus, Sum of coefficient = 0

(E) *(a) Both (A) and (R) are true and (R) is the correct explanation of (A).*

Explanation: Given: $(101)^4$

Here, 101 can be written as the sum or the difference of two numbers, such that the binomial theorem can be applied.

Therefore, 101 = 100 + 1

Hence, $(101)^4 = (100 + 1)^4$

Now, by applying the binomial theorem, we get

$(101)^4 = (100 + 1)^4 = {}^4C_0(100)^4 + {}^4C_1(100)^3(1) + {}^4C_2(100)^2(1)^2 + {}^4C_3(100)(1)^3 + {}^4C_4(1)^4$

$(101)^4 = (100)^4 + 4\,(100)^3 + 6(100)^2 + 4(100) + (1)^4$

$(101)^4 = 100000000 + 4000000 + 60000 + 400 + 1$

$(101)^4 = 104060401$

12. *Four friends applied the knowledge of Binomial Theorem while playing a game to make the equations by observing some conditions they make some equations.*

(A) Expand, $(1 - x + x^2)^4$.
(B) Expand the expression, $(1 - 3x)^7$.
(C) Show that $11^9 + 9^{11}$ is divisible by 10.

Ans. (A) We have,

$(1 - x + x^2)^4 = [(1 - x) + x^2]^4$

$= {}^4C_0(1 - x)^4 + {}^4C_1(1 - x)^3\,(x^2) + {}^4C_2(1 - x)^2\,(x^2)^2 + {}^4C_3(1 - x)\,(x^2)^3 + {}^4C_4(x^2)^4$

$= (1 - x)^4 + 4x^2\,(1 - x)^3 + 6x^4(1 - x)^2 + 4x^6\,(1 - x) + 1x^8$

$= (1 - 4x + 6x^2 - 4x^3 + x^4) + 4x^2(1 - 3x + 3x^2 - x^3) + 6x^4(1 - 2x + x^2) + 4(1 - x)x^6 + x^8$

$= 1 - 4x + 6x^2 - 4x^3 + x^4 + 4x^2 - 12x^3 + 12x^4 - 4x^5 + 6x^4 - 12x^5 + 6x^6 + 4x^6 - 4x^7 + x^8$

$= 1 - 4x + 10x^2 - 16x^3 + 19x^4 - 16x^5 + 10x^6 - 4x^7 + x^8$

(B) Here, $a = 1$, $b = 3x$, and $n = 7$

Given, $(1 - 3x)^7 = {}^7C_0(1)^7 - {}^7C_1(1)^6(3x)^1 + {}^7C_2(1)^5(3x)^2 - {}^7C_3(1)^1(3x)^3 + {}^7C_4(1)^3(3x)^4 - {}^7C_5(1)^2(3x)^5 + {}^7C_6(1)^1(3x)^6 - {}^7C_7(1)^0(3x)^7$

$= 1 - 21x + 189x^2 - 945x^3 + 2835x^4 - 5103x^5 + 5103x^6 - 2187x^7.$

(C) $11^9 + 9^{11} = (10 + 1)^9 + (10 - 1)^{11}$

$= ({}^9C_0.10^9 + {}^9C_1.10^8 + ... {}^9C_9) + ({}^{11}C_0.10^{11} - {}^{11}C_1.10^{10} + ... - {}^{11}C_{11})$

$= {}^9C_0.10^9 + {}^9C_1.10^8 + ... + {}^9C_8.10 + 1 + 10^{11} - {}^{11}C_1.10^{10} + ... + {}^{11}C_{10}.10 - 1$

$= 10[{}^9C_0.10^8 + {}^9C_1.10^7 + ... + {}^9C_8 + {}^{11}C_0.10^{10} - {}^{11}C_1.10^9 + ... + {}^{11}C_{10}]$

$= 10$ K, which is divisible by 10.

VERY SHORT ANSWER Type Questions (VSA)

[1 mark]

13. Find the coefficient of x in the expansion of $(1 - 3x + 7x^2)(1 - x)^{16}$. [NCERT Exemplar]

Ans. Given coefficient $(1 - 3x + 7x^2)(1 - x)^{16}$

We know that,

$T_{r+1} = {}^nC_r a^{n-r} b^r \; (1 - 3x + 7x^2)(1 - x)^{16}$

$= (1 - 3x + 7x^2)({}^{16}C_0 - {}^{16}C_1 x^1 + {}^{16}C_2 x^2 + ... + {}^{16}C_{16} x^{16})$

$= (1 - 3x + 7x^2)(1 - 16x + 120x^2 + ...)$

Coefficient of $x = -16 + 3 = -19$

14. Find the coefficient of x^{15} in the expansion of $(x - x^2)^{10}$. [NCERT Exemplar]

Ans. Let x^{15} occur in the $(r+1)^{th}$ term. Then,

$T_{r+1} = {}^{10}C_r x^{10-r}(-x^2)^2$

Given, $(x - x^2)^{10}$

$T_{r+1} = {}^{10}C_r x^{10-r}(-x^2)^r$

$= (-1)^r\, {}^{10}C_r x^{10-r} x^{2r}$

$= (-1)^r\, {}^{10}C_r x^{10+r}$

For the coefficient of x^{15}, we have

$10 + r = 15$

$\Rightarrow \quad r = 5$

$T_{5+1} = (-1)^5\, {}^{10}C_5 x^{15}$

Coefficient of

$$x^{15} = -\frac{10 \times 9 \times 8 \times 7 \times 6 \times 5!}{5 \times 4 \times 3 \times 2 \times 1 \times 5!} = -252$$

15. Find the 3rd term from beginning in the expansion of $(x^3 + 3a)^5$.

Ans. The 3rd term is

$= {}^5C_2 (x^3)^3 (3a)^2$

$= 10(x^9)\,9(a)^2$

16. Find the coefficient of x^n in expansions of $(1 + x)(1 - x)^n$. [Delhi Gov. QB 2022]

Ans. Coefficients of x^n in $(1 + x)(1 - x)^n$

$= (1 + x)({}^nC_0 - {}^nC_1.x + ... + (-1)^{n-1}.{}^nC_{n-1}x^{n-1} + (-1)^n\,{}^nC_n x^n)$

$= (-1)^n.{}^nC_n + (-1)^{n+1}.{}^nC_{n-1}$

$= (-1)^n(1 - n)$

17. Find the number of terms in the expansion $(1 - 3x + 3x^2 - x^3)^9$.

Ans. We have, $(1 - 3x + 3x^2 - x^3)^9$

$= ((1 - x)^3)^9 = (1 - x)^{27}$

$\therefore$ Total number of terms $= \dfrac{51+1}{2} = 26$

SHORT ANSWER Type-I Questions (SA-I)

[2 marks]

18. Using Binomial theorem, find the value of $(0.98)^{14}$ upto 4 places of decimal.

Ans. $(0.98)^{14} = (1 - 0.02)^{14}$

$= 1 + {}^{14}C_1(-0.02)^1 + {}^{14}C_2(-0.02)^2 + {}^{14}C_3(-0.02)^3$

[Neglecting higher powers of (0.01)]

$= 1 - 14(0.02) + 91(0.0004) - 364(0.000008)$

$= 1 - 0.28 + 0.0364 - 0.002912 = 0.753488.$

19. For what value of 'p', the coefficients of $(2p + 1)^{th}$ and $(4p + 5)^{th}$ terms in the expansion of $(1 + a)^{10}$ are equal?

Ans. Coeff. of $(2p + 1)^{th}$ term in $(1 + a)^{10} = {}^{10}C_{2p}$

And coeff. of $(4p + 5)^{th}$ term in $(1 + a)^{10}$

$= {}^{10}C_{4p+4}$.

By the question, ${}^{10}C_{2p} = {}^{10}C_{4p+4}$

thus, either $10 = 2p + (4p + 4)$ or $2p = 4p + 4$

$\Rightarrow \quad 6p = 6 \quad$ or $2p = -4$

$\Rightarrow \quad p = 1 \quad$ or $p = -2$.

But p can't be – ve.

Hence, $p = 1$.

20. Find the remainder left out when $8^{2n} - (62)^{2n+1}$ is divided by 9. [Delhi Gov. QB 2022]

Ans. $8^{2n} - (62)^{2n+1} = (1 + 63)^n - (63 - 1)^{2n+1}$

$= (1 + 63)^n + (1 - 63)^{2n+1}$

$= (1 + {}^nC_1\, 63 + {}^nC_2\, (63)^2 + \ldots + (63)^n + [1 - {}^{(2n+1)}C_1\, 63 + {}^{(2n+1)}C_2 (63)^2 + \ldots + (-1)(63)^{(2n+1)}]$

$= 2 + 63\, [{}^nC_1 + {}^nC_2 (63) + \ldots + (63)^{n-1} + {}^{(2n+1)}C_2(63) - \ldots - (63)^{(2n)}]$

Hence, the remainder is 2.

21. Expand $\left(y^2 + \frac{2}{y}\right)^5, y \neq 0$.

Ans. By using binomial theorem, we have

$\left(y^2 + \frac{2}{y}\right)^5$

$= {}^5C_0(y^2)^5 + {}^5C_1(y^2)^4\left(\frac{2}{y}\right)^1 + {}^5C_2(y^2)^3\left(\frac{2}{y}\right)^2 + {}^5C_3(y^2)^2\left(\frac{2}{y}\right)^3 + {}^5C_4(y^2)^1\left(\frac{2}{y}\right)^4 + {}^5C_5(y^2)^0\left(\frac{2}{y}\right)^5$

$\Rightarrow y^7 + 5y^8\left(\frac{2}{y}\right) + 10y^6\left(\frac{2}{y}\right)^2 + 10y^4\left(\frac{2}{y}\right)^3 + 5y^2\left(\frac{2}{y}\right)^4 + \left(\frac{2}{y}\right)^5$

$\Rightarrow y^7 + 10y^7 + 40y^4 + 80y + \frac{80}{y^2} + \frac{32}{y^5}$

22. Which is larger $(1.01)^{10,000.00}$ or 20,000.

Ans. $(1.01)^{10,00000} = (1 + 0.01)^{10,00000}$

$= {}^{10,0000}C_0 + {}^{10,00000}C_1\,(0.01)$ + other positive term

$= 1 + 10,00000 \times 0.01$

$= 10,001$

Hence, $(1.01)^{10,00000} < 20,000$.

SHORT ANSWER Type-II Questions (SA-II)

[3 marks]

23. Using binomial theorem, find the remainder when 5^{103} is divided by 13. [Delhi Gov. QB 2022]

Ans. We have, $5^{103} = 5.5^{102} = 5(26 - 1)^{51}$

Now expanding the above by binomial theorem we get,

$= [{}^{51}C_0(26)^{51}\, 1^0 - {}^{51}C_1(26)^{50}\, 1^1 + \ldots + {}^{51}C_{51}\, (26)^0\, 1^{51}]$

In the above expansion, all terms except the constant (last) term will contain 26, which is divisible by 13.

Hence, the remainder is

$5(-1)^{51} = 5$

Which is same as the remainder being 8.

24. Find out which one is larger $99^{50} + 100^{50}$ or 101^{50}. [Diksha]

Ans. Let's try to find out $101^{50} - 99^{50}$ in terms of remaining term *i.e.*

$101^{50} - 99^{50} = (100 + 1)^{50} - (100 - 1)^{50}$

$= (C_0.\, 100^{50} + C_1.\, 100^{49} + C_2.\, 100^{48} + \ldots)$

$= (C_0.\, 100^{50} - C_1.\, 100^{49} + C_2.\, 100^{48} \ldots)$

$= 2(C_1.\, 100^{49} + C_3.\, 100^{47} + \ldots)$

$= 2(50.\, 100^{49} + C_3.\, 100^{47} + \ldots)$

$= 100^{50} + 2(C_3.\, 100^{47} + \ldots)$

$= 100^{50}$

$\Rightarrow 101^{50} > 99^{50} + 100^{50}$

25. Evaluate $(102)^4$.

Ans. Given: $(102)^4$.

Here, 102 can be written as the sum or the difference of two number, such that the binomial theorem can be applied.

Therefore, $\quad 102 = 100 + 2$

Hence, $\quad (102)^4 = (100 + 2)^4$

Now, by applying binomial theorem, we get

$(102)^4 = (100 + 2)^4 = {}^4C_0(100)^4 + {}^4C_1(100)^3\,(2) + {}^4C_2(100)^2\,(2)^2 + {}^4C_3(100)^1\,(2)^3 + {}^4C_4(2)^4$

$= (100)^4 + 8(100)^3 + 24(100)^2 + 32(100) + 16$

$= 100000000 + 8000000 + 240000 + 3200 + 16$

$= 108243216$

26. Evaluate $(\sqrt{2}+1)^5-(\sqrt{2}-1)^6$ using binomial theorem. [Delhi Gov. QB 2022]

Ans. $(\sqrt{2}+1)^5-(\sqrt{2}-1)^5$

$$[{}^5C_0(\sqrt{2})^5+{}^5C_1(\sqrt{2})^4(1)+{}^5C_2(\sqrt{2})^3(1)^2+{}^5C_3(\sqrt{2})^2(1)^3+{}^5C_4(\sqrt{2})(1)^4+{}^5C_5(1)^5]$$
$$-[{}^5C_0(\sqrt{2})^5+{}^5C_1(\sqrt{2})^4(-1)+{}^5C_2(\sqrt{2})^3(-1)^2+{}^5C_3(\sqrt{2})^2(-1)^3+{}^5C_4(\sqrt{2})(-1)^4+{}^5C_5(-1)^5]$$

$= 2[{}^5C_1(\sqrt{2})^4+{}^5C_3(\sqrt{2})^2+{}^5C_5]$

$= 2\left[5\times 4+\frac{5!}{3!2!}\times 2+1\right]$

$= 2\left[20+\frac{5\times 4\times 3!}{3!\times 2}\times 2+1\right]$

$= 2\ (21 + 20)$

$= 2 \times 41$

$= 82$

27. Using binomial theorem, expand

$$(x + y)^6 - (x - y)^6.$$

Hence, find the value of

$$\left(\sqrt{3}+1\right)^6-\left(\sqrt{3}-1\right)^6.$$

Ans. $(x+y)^6-(x-y)^6 = {}^6C_0x^6+{}^6C_1x^5y+{}^6C_2x^4y^2+{}^6C_3x^3y^3+{}^6C_4x^2y^4+{}^6C_5xy^5+{}^6C_6x^0y^6$
$-[{}^6C_0x^6+{}^6C_1x^5(-y)^1+{}^6C_2x^4(-y)^2+{}^6C_3x^3(-y)^3+{}^6C_4x^2(-y)^4+{}^6C_5x(-y)^5+{}^6C_6x^0(-y)^6]$

$= 2(6x^5y + 20x^3y^3 + 6xy^5)$

$= 4xy(3x^4 + 10x^2y^2 + 3y^4)$

On substituting $x = \sqrt{3}$ and $y = 1$, we get

$= 4\times\sqrt{3}\times 1\left(3\left(\sqrt{3}\right)^4+10(\sqrt{3})^2(1)^2+3(1)^4\right)$

$= 4\sqrt{3}(3\times 9+10\times 3+3)$

$= 4\sqrt{3}\,(27+30+3)$

$= 4\sqrt{3}\,(60)$

$= 240\sqrt{3}$

LONG ANSWER Type Questions (LA)

[**4** & **5** marks]

28. Using the binomial theorem, show that $6^n - 5n$ always leaves remainder 1 when divided by 25.

Ans. For any two numbers, say a and b, we can find numbers x and y such that $a = bx + y$, then we say that c divides a with x as quotient and y as remainder. Thus, in over to show that $6^n - 5n$ leaves remainder 1 when divided by 25, we should prove that $6^n - 50 = 25k + 1$, where K is some natural number.

We know that,

$$(1+a)^n = {}^nC_0 + {}^nC_1\,a + {}^nC_2\,a^2 + \ldots + {}^nC_n\,a^n$$

Now, for $a = 5$, we get

$$(1+5)^n = {}^nC_0 + {}^nC_1\,5 + {}^nC_2\,(5)^2 + \ldots + {}^nC_n\,(5)^n$$

Now the above from can be written as :

$$6^n = 1 + 5n + 5^2\,{}^nC_2 + 5^3\,{}^nC_3 + \ldots + 5^n$$

Now, bring $5n$ to the L.H.S., we get

$$6^n - 5n = 1 + 5^2\,{}^nC_2 + 5^3\,{}^nC_3 + \ldots + 5^n$$
$$6^n - 5n = 1 + 5^2({}^nC_2 + 5\,{}^nC_3 + \ldots + 5^{n-2})$$
$$6^n - 5n = 1 + 25\,({}^nC_2 + 5\,{}^nC_3 + \ldots + 5^{n-2})$$
$$6^n - 5n = 1 + 25\,K$$

(Where $K = {}^nC_2 + 5\,{}^nC_3 + \ldots + 5^{n-2}$)

Hence, proved.

29. Using binomial theorem, expand the following expansion.

(A) $(x^2 + 2a)^4$

(B) $\left(2+\sqrt{3}\right)^7+\left(2-\sqrt{3}\right)^7$

(C) $\left(x^2+3+2\sqrt{3}x\right)^3$

(D) $(x^2 - 2x + 1)^3$

Ans. (A) Here, $a = x^2$, $b = 2a$ and $n = 4$

Given, $(x^2+2a)^4 = {}^4C_0(x^2)^4 + {}^4C_1(x^2)^3(2a)^1 + {}^4C_2(x^2)^2(2a)^2 + {}^4C_3(x^2)^1(2a)^3 + {}^4C_4(x^2)^0(2a)^4$

$= 1(x^8) + 4(x^6)(2a) + 6x^4\,4a^2 + 4x^2\,8a^3 + 16a^4$

$= x^8 + 8x^6a + 24x^4a^2 + 32x^2a^3 + 16a^4$

(B) $\left(2+\sqrt{3}\right)^7$

$= {}^7C_0(2)^7\times(\sqrt{3})^0 + {}^7C_1(2)^6\times(\sqrt{3})^1 + {}^7C_2(2)^5(\sqrt{3})^2 + {}^7C_3(2)^4(\sqrt{3})^3 + {}^7C_4(2)^3(\sqrt{3})^4 + {}^7C_5(2)^2(\sqrt{3})^5 + {}^7C_6(2)^1(\sqrt{3})^6 + {}^7C_7(2)^0(\sqrt{3})^7$

and $\left(2-\sqrt{3}\right)^7$

$= {}^7C_0(2)^7(\sqrt{3})^0 + {}^7C_1(2)^6(\sqrt{3})^1 - {}^7C_2(2)^5$

$(\sqrt{3})^2 - {}^7C_3(2)^4(\sqrt{3})^3 + {}^7C_4(2)^3(\sqrt{3})^4$

$- {}^7C_5(2)^2(\sqrt{3})^5 + {}^7C_6(2)^1(\sqrt{3})^6$

$- {}^7C_7(2)^0(\sqrt{3})^7.$

Now, $(2+\sqrt{3})^7 + (2-\sqrt{3})^7$ we get,

$= 2[{}^7C_0(2)^7(\sqrt{3})^0 + {}^7C_2(2)^5(\sqrt{3})^2 + {}^7C_4(2)^3(\sqrt{3})^4 + {}^7C_6(2)^1(\sqrt{3})^6]$

$= 10084$

(C) $\left(x^2+3+2\sqrt{3}x\right)^3 = \left(x+\sqrt{3}\right)^6$

$= {}^6C_0(x)^6 \times (\sqrt{3})^0 + {}^6C_1\, x^5(\sqrt{3})^1 + {}^6C_2\, x^4$

$(\sqrt{3})^2 + {}^6C_3\, x^3(\sqrt{3})^3 + {}^6C_4\, x^2(\sqrt{3})^4 +$

${}^6C_5\, x^1(\sqrt{3})^5 + {}^6C_6(\sqrt{3})^6$

$= x^6 + 6\sqrt{3}x^5 + 45x^4 + 60\sqrt{3}\,x^3 +$

$135x^2 + 54\sqrt{3}x + 27.$

(D) We have, $(x^2 - 2x + 1)^3 = [(x-1)^2]^3 = (x-1)^6$

$= {}^6C_0(x)^6 - {}^6C_1(x^5)(1) + {}^6C_2(x)^4$

$- {}^6C_3(x)^3 + {}^6C_4(x)^2 - {}^6C_5(x)^1 + {}^6C_6(1)^6$

$= x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1.$

30. Show that the coefficient of x^5 in the expansion of product $(1+2x)^6(1-x)^7$ is 171.

[Delhi Gov. QB 2022]

Ans. Coefficient of x^5 in the product of $(1+2x^6)(1-x^7)$

$(1+2x)^6(1-x)^7$

Expanding using Binomial theorem

$1 + {}^6C_1(2x) + {}^6C_2(2x)^2 + {}^6C_3(2x)^3 + + {}^6C_4(2x)^4$

$+ {}^6C_5(2x)^5 + + {}^6C_6(2x)^6] \times [1 - {}^7C_1x + {}^7C_2x^2$

$- {}^7C_3x^3 + {}^7C_4x^4 - {}^7C_5x^5 ....]$

$= [1 + 12x + 60x^2 + 160x^2 + 240x^4 + 192x^5$

$+ 64x^6] \times [1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 ....]$

Now, the coefficient of x^5 in the product is

$= 1 \times (-21) + 12 \times 35 + 60 \times (-35) + 160 \times 21$

$+ 240 \times (-7) + 192 \times 1$

$= -21 + 420 - 2100 + 3360 - 1680 + 192$

$= 171$

Hence, proved.